Debiasing Nonlinear Transformations Involving Correlated Measurement Components

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***Abstract*—Given a measurement value that has been corrupted with additive multivariate Gaussian noise with nonzero correlation in its covariance matrix, this paper derives two new Taylor series approximations to estimating an unbiased mean and a consistent covariance matrix. The conversion produces mean and covariance estimates that are more consistent than other expansions in the literature when the covariance matrix is not a diagonal matrix.**

I. Introduction

When performing target tracking, there often arises the need to perform a nonlinear transformation of a multivariate Gaussian random vector and approximate the transformed result as being Gaussian distributed. One of the most common instances where this arises is when converting measurements from one coordinate system to another. It has long been realized that if $\mathbf{f}(\mathbf{z})$ performs a transformation of the deterministic vector $\mathbf{z}$, and one uses a noisy $\hat{\mathbf{z}}$ instead of $\mathbf{z}$, then quite often, the result is biased. That is,

$$\mathrm{E}\{\mathbf{f}(\hat{\mathbf{z}})\} \neq \mathbf{f}(\mathbf{z}) \tag{1}$$

where $\mathrm{E}\{\cdot\}$ is the expected value. As a result, many attempts have been made over the years to slightly modify the transformation $\mathbf{f}$ into $\hat{\mathbf{f}}$ such that

$$\mathrm{E}\left\{\hat{\mathbf{f}}(\hat{\mathbf{z}})\right\} = \mathbf{f}(\mathbf{z}) \tag{2}$$

and one obtains an accurate associated covariance matrix of the debiased estimator. For example, methods for debiased conversion of polar and/or spherical measurements are described in [9], [10], [1, Ch. 10.4], and [5]. Similarly debiased conversion techniques for monostatic range and direction cosine are given in [15] and [3]. Such techniques have been applied for the unbiased fusion of two spherical direction measurements to obtain a position and an associated covariance matrix in [6] and [7].

Most of the aforementioned techniques are based on performing some sort of a Taylor series expansion. However, a challenge with utilizing a technique based on a Taylor series expansion of the nonlinear transformation is that one typically does not have the true derivative values required (because of measurement noise). More recently, a series of papers have looked at Cartesian measurement conversion [14], [13] and [12]. These papers have noted that the traditional approach to obtaining a debiased converted measurement and covariance matrix as in [2], does not work as well as one might hope. For second order Taylor series expansions for measurement conversion, the innovation of [14], [13], and [12] is to perform two Taylor series expansions: A second order expansion with respect to $\mathbf{f}$ and a first order expansions with respect to the Jacobian of $\mathbf{f}$. That allows one to reduce the error associated with evaluating the first derivatives at the noisy point $\hat{\mathbf{z}}$ instead of at $\mathbf{z}$.

The general measurement conversion approach described in [12] works well when the components of $\hat{\mathbf{z}}$ are uncorrelated. However, the expression given when the measurement noise is correlated will be shown to perform poorly in some instances. This article rederives the second-order Taylor method of [13] and [12] in Section II, including some omitted terms in a Taylor series expansion of some derivative terms, providing a superior solution for the case where the noise between the measurement components is correlated. Specifically, two different approaches are introduced. Section III then shows that the second of these generalized conversion methods reduces to the diagonal solution of [13] if the measurement components are uncorrelated.

Section IV compares the Taylor-series methods of this paper with that of Marom in [12] as well as with a cubature integration technique as described in [4]. The results are concluded in Section V.

II. The General Conversion

Consider a d_z-dimensional noisy measurement vector $\hat{\mathbf{z}}$. The state being estimated is $d_x \times 1$. The relationship between the measurement if it were noise-free, $\mathbf{z}$, and the noisy measurement is

$$\hat{\mathbf{z}} = \mathbf{z} + \mathbf{w} \tag{3}$$

where $\mathbf{w}$ is a $d_z \times 1$ Gaussian random variable with zero mean and covariance matrix $\mathbf{R}$. The element in row i and column j of $\mathbf{R}$ is designated R_{ij}. We are given a transformation

$$\mathbf{x} = \mathbf{f}(\mathbf{z}) \tag{4}$$

In the presence of a noisy measurement, a biased estimator of $\mathbf{x}$ is

$$\hat{\mathbf{x}} = \mathbf{f}(\hat{\mathbf{z}}) = \mathbf{f}(\mathbf{z} + \mathbf{w}) \tag{5}$$

The basic approach to obtaining a debiased estimate of $\hat{\mathbf{x}}$ is to perform a Taylor series expansion of (5) about the $\mathbf{w} = 0$ point and then adjust $\hat{\mathbf{x}}$ so that the expected value is zero. A fourth-order Taylor series expansion of the ath component of $\hat{\mathbf{x}}$ about the $\mathbf{w} = \mathbf{0}$ point is [8, Ch. 2.4]:

$$\begin{aligned}\hat{x}_a \approx & f_a(\mathbf{z}) + \sum_{j=1}^{d_z} \frac{\partial f_a(\mathbf{z})}{\partial z_j} w_j + \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} \frac{\partial^2 f_a(\mathbf{z})}{\partial z_j \partial z_k} w_j w_k \\ & + \frac{1}{6}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z} \frac{\partial^3 f_a(\mathbf{z})}{\partial z_j \partial z_k \partial z_l} w_j w_k w_l \\ & + \frac{1}{24}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\sum_{m=1}^{d_z} \frac{\partial^4 f_a(\mathbf{z})}{\partial z_j \partial z_k \partial z_l \partial z_m} w_j w_k w_l w_m \quad (6)\end{aligned}$$

where f_a is the ath component of $\mathbf{f}$ and the subscripts of z and w indicate the components of $\mathbf{z}$ and $\mathbf{w}$ that are used. Note that the derivatives in (6) are taken with respect to elements of $\mathbf{z}$, which we do not have, and not with respect to elements of the measurement $\hat{\mathbf{z}}$. Though (given noise-free derivatives) higher-order expansions will be increasingly accurate, as in previous work, the focus is on first and second order Taylor-series expansion.

A. The First Order Expansion

Expressions for a first order expansion are well known in the literature and offer one of the simplest techniques for obtaining a covariance matrix when performing a simple transformation of a Gaussian random variable. They are reviewed here and can serve as a baseline. In the case of a first-order expansion, the final three terms of (6) are removed to get

$$\hat{x}_a \approx f_a(\mathbf{z}) + \sum_{j=1}^{d_z} \frac{\partial f_a(\mathbf{z})}{\partial z_j} w_j \quad (7)$$

With only a first order expansion, $\hat{x}_a$ is unbiased

$$\mathrm{E}\left\{\hat{x}_a\right\} = f_a(\mathbf{z}) \quad (8)$$

Thus, the first order estimator is just

$$x_a^{\text{first order}} = \hat{x}_a \quad (9)$$

For simplicity of notation, the first derivative shall be abbreviated as

$$J_{a,j} = \frac{\partial f_a(\mathbf{z})}{\partial z_j} \quad (10)$$

The value of $J_{a,j}$ is not actually available. What is available is the gradient evaluated at the measurement

$$\hat{J}_{a,j} = \frac{\partial f_a(\hat{\mathbf{z}})}{\partial z_j} \quad (11)$$

Thus, for the first order method, we approximate

$$J_{a,j} \approx \hat{J}_{a,j} \quad (12)$$

Define the element in row a and column b of the covariance of $\hat{x}_a$ as

$$R_{ab}^{\text{first order}} = \mathrm{E}\left\{\left(x_a^{\text{first order}} - \mathrm{E}\left\{x_a^{\text{first order}}\right\}\right)\left(x_b^{\text{first order}} - \mathrm{E}\left\{x_b^{\text{first order}}\right\}\right)\right\} \quad (13)$$

Using the first-order Taylor series approximation, the difference needed for the covariance expression is

$$x_a^{\text{first order}} - \mathrm{E}\left\{x_a^{\text{first order}}\right\} = \sum_{j=1}^{d_z} \hat{J}_{a,j} w_j \quad (14)$$

Consequently, the value of $R_{ab}^{\text{first order}}$ simplifies to

$$R_{ab}^{\text{first order}} = \sum_{j=1}^{d_z}\sum_{j=1}^{d_z} \hat{J}_{a,j}\hat{J}_{b,k} R_{j,k} \quad (15)$$

B. The Second Order Expansion

In the case of a second-order expansion, only the final two terms of (6) are removed.

$$\hat{x}_a \approx f_a(\mathbf{z}) + \sum_{j=1}^{d_z} \frac{\partial f_a(\mathbf{z})}{\partial z_j} w_j + \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} \frac{\partial^2 f_a(\mathbf{z})}{\partial z_j \partial z_k} w_j w_k \quad (16)$$

For simplicity of notation, the first derivatives shall be abbreviated as in (10) and the second derivatives with

$$H_{a,j,k} = \frac{\partial^2 f_a(\mathbf{z})}{\partial z_j \partial z_k} \quad (17)$$

So the expansion of (16) can be written.

$$\hat{x}_a = f_a(\mathbf{z}) + \sum_{j=1}^{d_z} J_{a,j} w_j + \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} H_{a,j,k} w_j w_k \quad (18)$$

Because the noise-free measurement $\mathbf{z}$ is not available, $J_{a,j}$ and $H_{a,j,k}$ cannot be computed. Rather, the first and second derivatives can be evaluated at the noisy measurement value:

$$\hat{J}_{a,j} = \frac{\partial f_a(\hat{\mathbf{z}})}{\partial z_j} \qquad \hat{H}_{a,j,k} = \frac{\partial^2 f_a(\hat{\mathbf{z}})}{\partial z_j \partial z_k} \quad (19)$$

The "secret sauce" in the measurement conversion algorithm of [12] is to realize that just as a Taylor series was used to expand $\hat{x}_a$ as in (16), so too can one expand $\hat{J}_{a,j}$. Assuming that we do not want to evaluate higher than second derivatives of f_a, this means that we shall make the approximation that

$$H_{a,j,k} \approx \hat{H}_{a,j,k} \quad (20)$$

Additionally, a first-order Taylor-series expansion of $\hat{J}_{a,j}$ about the $\mathbf{w} = \mathbf{0}$ point can be performed, leading to

$$\begin{aligned}\hat{J}_{a,j} \approx & J_{a,j} + \sum_{k=1}^{d_z} \frac{\partial J_{a,j}}{\partial z_k} w_k \quad (21) \\ = & J_{a,j} + \sum_{k=1}^{d_z} H_{a,j,k} w_k \quad (22)\end{aligned}$$

Note that in [14], [13], and [12], the Taylor series expansion of $\hat{J}_{a,j}$ omitted all of the cross terms. That is, they used $\hat{J}_{a,j} \approx J_{a,j} + H_{a,j,j} w_j$. This paper derives the solution including all of the cross terms, and demonstrates that when $\mathbf{R}$ is not diagonal, the performance of the debiasing using all of the cross terms is superior.

Using the approximation in (20) and rearranging (22) leads to an approximation for $J_{a,j}$ of

$$J_{a,j} \approx \hat{J}_{a,j} - \sum_{k=1}^{d_z} \hat{H}_{a,j,k} w_k \tag{23}$$

At this point, a decision can be made as to where to apply the approximation in (23). Two approaches are considered:

1) Substitute (23) and (20) for J and H into (18) and then derive a debiased estimator.
2) Derive a debiased estimator assuming that J and H are perfectly known and then substitute (23) and (20) for J and H. This is the substitution that is made in [13].

Both techniques shall be considered and are shown to produce opposite values for the debiasing term.

Considering the first approach, substituting (23) and (20) into (18) leads to

$$\hat{x}_a^1 = f_a(\mathbf{z}) + \sum_{j=1}^{d_z} \hat{J}_{a,j} w_j - \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} w_j w_k \tag{24}$$

where the superscript of 1 indicates that this is the first debiasing technique. Note that this is the same as (18), except the gradients are evaluated at the measurement values and the sign of the second term has been flipped.

The expected value of the biased estimator $\hat{x}_a^1$ can thus be approximated as

$$\mathrm{E}\left\{\hat{x}_a^1\right\} = f_a(\mathbf{z}) - \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} R_{j,k} \tag{25}$$

since $\mathbf{w}$ is zero mean with covariance matrix $\mathbf{R}$. Consequently, a debiased estimator subtracts away the bias terms. The debiased estimator is

$$x_a^{1,\text{debiased}} = \hat{x}_a + \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} R_{j,k} \tag{26}$$

The sign before the sum in (26) is positive.

Next, we consider the second approach to obtaining a debiased estimator. In this instance, we begin by taking the expected value of (18) to get

$$\mathrm{E}\left\{\hat{x}_a^2\right\} = f_a(\mathbf{z}) + \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} H_{a,j,k} R_{j,k} \tag{27}$$

which, using the approximation of (20), leads to a debiased estimator of

$$x_a^{2,\text{debiased}} = \hat{x}_a - \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} R_{j,k} \tag{28}$$

whose bias correction has the opposite sign of (26)!

In the following two subsections, covariance matrices for $x_a^{1,\text{debiased}}$ and $x_a^{2,\text{debiased}}$ are derived. In the simulations of Section IV, it is shown that while the covariance matrix for each "debiased" estimate is quite consistent, the performance is not identical.

C. Covariance Derivation for the First Second-Order Debiasing Method

Using the second-order Taylor series approximation of (24) and the expectation of (26) a difference needed for approximating the covariance matrix of the estimate is

$$\begin{aligned} x_a^{1,\text{debiased}} - \mathrm{E}\left\{x_a^{1,\text{debiased}}\right\} &= \sum_{j=1}^{d_z} \hat{J}_{a,j} w_j \\ - \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} w_j w_k &+ \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} R_{j,k} \qquad (29) \\ = \sum_{j=1}^{d_z} \hat{J}_{a,j} w_j &+ \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} \left(R_{j,k} - w_j w_k\right) \qquad (30) \end{aligned}$$

Define the element in row a and column b of the covariance of x_a^{debiased} as

$$\begin{aligned} R_{ab}^{1,\text{debiased}} \triangleq \mathrm{E}\Big\{ &\left(x_a^{1,\text{debiased}} - \mathrm{E}\left\{x_a^{1,\text{debiased}}\right\}\right) \\ &\cdot \left(x_b^{1,\text{debiased}} - \mathrm{E}\left\{x_b^{1,\text{debiased}}\right\}\right)\Big\} \end{aligned} \tag{31}$$

Using (30), this simplifies to

$$\begin{aligned} R_{ab}^{1,\text{debiased}} = \mathrm{E}\Bigg\{ &\left(\sum_{j=1}^{d_z} \hat{J}_{a,j} w_j + \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{a,j,k} \left(R_{j,k} - w_j w_k\right)\right) \\ &\cdot \left(\sum_{j=1}^{d_z} \hat{J}_{b,j} w_j + \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{H}_{b,j,k} \left(R_{j,k} - w_j w_k\right)\right)\Bigg\} \end{aligned} \tag{32}$$

$$\begin{aligned} &= \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \hat{J}_{a,j} \hat{J}_{b,k} \mathrm{E}\left\{w_j w_k\right\} \\ &+ \frac{1}{2} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \sum_{l=1}^{d_z} \left(\hat{J}_{a,l} \hat{H}_{b,j,k} + \hat{J}_{b,l} \hat{H}_{a,j,k}\right) \mathrm{E}\left\{\left(R_{j,k} - w_j w_k\right) w_l\right\} \\ &+ \frac{1}{4} \sum_{j=1}^{d_z} \sum_{k=1}^{d_z} \sum_{l=1}^{d_z} \sum_{m=1}^{d_z} \hat{H}_{b,l,m} \hat{H}_{a,j,k} \mathrm{E}\left\{\left(R_{j,k} - w_j w_k\right)\left(R_{l,m} - w_l w_m\right)\right\} \end{aligned} \tag{33}$$

The expected values can be simplified as

$$\begin{aligned} &\mathrm{E}\left\{w_j w_k\right\} = R_{jk} && (34) \\ &\mathrm{E}\left\{\left(R_{j,k} - w_j w_k\right) w_l\right\} = 0 && (35) \\ &\mathrm{E}\left\{\left(R_{j,k} - w_j w_k\right)\left(R_{l,m} - w_l w_m\right)\right\} = R_{j,k} R_{l,m} \\ &- R_{j,k} \mathrm{E}\left\{w_l w_m\right\} - R_{l,m}\left\{w_j w_k\right\} + \mathrm{E}\left\{w_j w_k w_l w_m\right\} && (36) \\ &= \mathrm{E}\left\{w_j w_k w_l w_m\right\} - R_{j,k} R_{l,m} && (37) \end{aligned}$$

where (34) comes from the definition of the covariance when $\mathbf{w}$ is zero-mean and (35) come from the fact that all odd moments of normal random vectors are zero.

The fact that all off moments are zero as well as expressions for the fourth-order moments $\mathrm{E}\left\{w_j w_k w_l w_m\right\}$ can be obtained by evaluating the moment generating function of a multivariate Gaussian random variable, as described in [1, Ch. 1.4.8]. The expectation $\mathrm{E}\left\{w_j w_k w_l w_m\right\}$ depends on how many of the terms are equal. Specifically, for a generic set of all unique a, b, c, d subscripts, one can write

$$\mathrm{E}\left\{w_a^4\right\} = 3R_{aa}^2 \tag{38}$$

$$\mathrm{E}\left\{w_a^3 w_b\right\} = 3R_{aa}R_{ab} \tag{39}$$

$$\mathrm{E}\left\{w_a^2 w_b^2\right\} = 2R_{ab}^2 + R_{aa}R_{bb} \tag{40}$$

$$\mathrm{E}\left\{w_a^2 w_b w_c\right\} = 2R_{ab}R_{ac} + R_{aa}R_{bc} \tag{41}$$

$$\mathrm{E}\left\{w_a w_b w_c w_d\right\} = R_{ad}R_{bc} + R_{ac}R_{bd} + R_{ab}R_{cd} \tag{42}$$

Thus, (33) becomes

$$\begin{aligned} R_{ab}^{1,\text{debiased}} = &\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} \hat{J}_{a,j}\hat{J}_{b,k}R_{j,k} \\ &- \frac{1}{4}\left(\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} \hat{H}_{a,j,k}R_{j,k}\right)\left(\sum_{j=1}^{d_z}\sum_{j=1}^{d_z} \hat{H}_{b,j,k}R_{j,k}\right) \\ &+ \frac{1}{4}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\sum_{m=1}^{d_z} \hat{H}_{b,l,m}\hat{H}_{a,j,k}c_{j,k,l,m} \end{aligned} \tag{43}$$

where $c_{j,k,l,m}$ is broken into 15 cases (the Bell number of 4 [16]), one for each set partition of the indices in $\mathrm{E}\{w_j w_k w_l w_m\}$. All cases simplify to

$$c_{j,k,l,m} = R_{j,m}R_{k,l} + R_{j,l}R_{k,m} + R_{j,k}R_{l,m} \tag{44}$$

Consequently, if one wishes to approximate $\mathbf{x}$ given $\hat{\mathbf{z}}$ as a multivariate Gaussian random variable using the first debiasing technique, then the mean is given by (26) and the covariance matrix is given by (43).

D. Covariance Derivation for the Second Second-Order Debiasing Method

Using the debiased estimator of (28) and the original Taylor series expansion of (18), the difference needed for approximating the covariance matrix is

$$\begin{aligned} x_a^{2,\text{debiased}} - \mathrm{E}\left\{x_a^{2,\text{debiased}}\right\} = &\sum_{j=1}^{d_z} J_{a,j}w_j \\ + \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} H_{a,j,k}w_j w_k &- \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z} \hat{H}_{a,j,k}R_{j,k} \end{aligned} \tag{45}$$

It is at this point that the approximations in (23) and (20) are substituted to get

$$\begin{aligned} x_a^{2,\text{debiased}} - \mathrm{E}\left\{x_a^{2,\text{debiased}}\right\} &= \sum_{j=1}^{d_z}\left(\hat{J}_{a,j} - \sum_{k=1}^{d_z}\hat{H}_{a,j,k}w_k\right)w_j \\ + \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{a,j,k}w_j w_k &- \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{a,j,k}R_{j,k} \end{aligned} \tag{46}$$

$$= \sum_{j=1}^{d_z}\hat{J}_{a,j}w_j - \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{a,j,k}\left(R_{j,k} + w_k w_j\right) \tag{47}$$

which is the same as (30) but with the sign in front of the $R_{j,k}$ term flipped. The covariance matrix is consequently

$$R_{ab}^{2,\text{debiased}} = \mathrm{E}\left\{\left(x_a^{2,\text{debiased}} - \mathrm{E}\left\{x_a^{2,\text{debiased}}\right\}\right)\left(x_b^{2,\text{debiased}} - \mathrm{E}\left\{x_b^{2,\text{debiased}}\right\}\right)\right\} \tag{48}$$

$$\begin{aligned} = \mathrm{E}\Bigg\{ &\left(\sum_{j=1}^{d_z}\hat{J}_{a,j}w_j - \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{a,j,k}\left(R_{j,k} + w_k w_j\right)\right) \\ \cdot &\left(\sum_{j=1}^{d_z}\hat{J}_{b,j}w_j - \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{b,j,k}\left(R_{j,k} + w_k w_j\right)\right)\Bigg\} \end{aligned} \tag{49}$$

$$\begin{aligned} &= \sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{J}_{a,j}\hat{J}_{b,k}\,\mathrm{E}\left\{w_j w_k\right\} \\ &- \frac{1}{2}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\left(\hat{J}_{a,l}\hat{H}_{b,j,k} + \hat{J}_{b,l}\hat{H}_{a,j,k}\right)\mathrm{E}\left\{\left(R_{j,k} + w_j w_k\right)w_l\right\} \\ &+ \frac{1}{4}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\sum_{m=1}^{d_z}\hat{H}_{b,l,m}\hat{H}_{a,j,k}\,\mathrm{E}\left\{\left(R_{j,k} + w_j w_k\right)\left(R_{l,m} + w_l w_m\right)\right\} \end{aligned} \tag{50}$$

Using (34) and

$$\mathrm{E}\left\{\left(R_{j,k} + w_j w_k\right)w_l\right\} = 0 \tag{51}$$

$$\begin{aligned} \mathrm{E}\left\{\left(R_{j,k} + w_j w_k\right)\left(R_{l,m} + w_l w_m\right)\right\} = &3R_{j,k}R_{l,m} \\ &+ \mathrm{E}\left\{w_j w_k w_l w_m\right\} \end{aligned} \tag{52}$$

equation (50) simplifies to

$$\begin{aligned} R_{ab}^{2,\text{debiased}} = &\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{J}_{a,j}\hat{J}_{b,k}R_{j,k} \\ &+ \frac{3}{4}\left(\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{a,j,k}R_{j,k}\right)\left(\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\hat{H}_{b,j,k}R_{j,k}\right) \\ &+ \frac{1}{4}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\sum_{m=1}^{d_z}\hat{H}_{b,l,m}\hat{H}_{a,j,k}c_{j,k,l,m} \end{aligned} \tag{53}$$

where $c_{j,k,l,m}$ is given by (44).

Consequently, if one wishes to approximate $\mathbf{x}$ given $\hat{\mathbf{z}}$ as a multivariate Gaussian random variable using the first debiasing technique, then the mean is given by (28) and the covariance matrix is given by (53).

III. Diagonal Measurement Covariance Matrix Simplification

If the covariance matrix $\mathbf{R}$ is diagonal, then (26) and (28) become

$$x_a^{1,\text{debiased}} = \hat{x}_a + \frac{1}{2}\sum_{j=1}^{d_z}\hat{H}_{a,j,j}R_{j,j} \tag{54}$$

$$x_a^{2,\text{debiased}} = \hat{x}_a - \frac{1}{2}\sum_{j=1}^{d_z}\hat{H}_{a,j,j}R_{j,j} \tag{55}$$

To simplify the covariance matrices, we note that if the cross terms are zero, then

$$c_{j,k,l,m} = \begin{cases} 3R_{j,j}^2 & \text{If } j=k=l=m \\ R_{j,j}R_{l,l} & \text{If } j=k \text{ and } l=m\neq j \\ R_{j,j}R_{m,m} & \text{If } j=l \text{ and } m=k\neq j \\ R_{j,j}R_{k,k} & \text{If } j=m \text{ and } k=l\neq j \\ 0 & \text{Otherwise} \end{cases} \tag{56}$$

Utilizing the identity that $H_{a,j,k} = H_{a,k,j}$, this means that

$$\begin{aligned}\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\sum_{l=1}^{d_z}\sum_{m=1}^{d_z}\hat{H}_{b,l,m}\hat{H}_{a,j,k}c_{j,k,l,m} = 3\sum_{j=1}^{d_z}\hat{H}_{a,j,j}\hat{H}_{b,j,j}R_{j,j}^2 \\ +\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\bar{\delta}_{j,k}\left(\hat{H}_{a,j,j}\hat{H}_{b,k,k}+2\hat{H}_{a,j,k}\hat{H}_{b,j,k}\right)R_{j,j}R_{k,k}\end{aligned} \tag{57}$$

where $\bar{\delta}_{j,k}$ is one minus the Kronecker delta function:

$$\bar{\delta}_{j,k} \triangleq \begin{cases} 0 & \text{If } j=k \\ 1 & \text{Otherwise} \end{cases} \tag{58}$$

Consequently, for a diagonal $\mathbf{R}$ matrix, the covariance matrices simplify to

$$\begin{aligned} R_{ab}^{1,\text{debiased}} &= \sum_{j=1}^{d_z}\hat{J}_{a,j}\hat{J}_{b,j}R_{j,j} \\ &-\frac{1}{4}\left(\sum_{j=1}^{d_z}\hat{H}_{a,j,j}R_{j,j}\right)\left(\sum_{j=1}^{d_z}\hat{H}_{b,j,j}R_{j,j}\right) \\ &+\frac{3}{4}\sum_{j=1}^{d_z}\hat{H}_{a,j,j}\hat{H}_{b,j,j}R_{j,j}^2 \\ &+\frac{1}{4}\sum_{j=1}^{dz}\sum_{k=1}^{dz}\bar{\delta}_{j,k}\left(\hat{H}_{a,j,j}\hat{H}_{b,k,k}+2\hat{H}_{a,j,k}\hat{H}_{b,j,k}\right)R_{j,j}R_{k,k}\end{aligned} \tag{59}$$

and

$$\begin{aligned} R_{ab}^{2,\text{debiased}} &= \sum_{j=1}^{d_z}\hat{J}_{a,j}\hat{J}_{b,j}R_{j,j} \\ &+\frac{3}{4}\left(\sum_{j=1}^{d_z}\hat{H}_{a,j,j}R_{j,j}\right)\left(\sum_{j=1}^{d_z}\hat{H}_{b,j,j}R_{j,j}\right) \\ &+\frac{3}{4}\sum_{j=1}^{d_z}\hat{H}_{a,j,j}\hat{H}_{b,j,j}R_{j,j}^2 \\ &+\frac{1}{4}\sum_{j=1}^{dz}\sum_{k=1}^{dz}\bar{\delta}_{j,k}\left(\hat{H}_{a,j,j}\hat{H}_{b,k,k}+2\hat{H}_{a,j,k}\hat{H}_{b,j,k}\right)R_{j,j}R_{k,k}\end{aligned} \tag{60}$$

Equation (60) is actually equivalent to Equation 41 in [13], which is derived with only a diagonal expansion of J (Drop all but the $k=j$ index of the sum in (23)). To see this, we apply the identity from Equation 32 of [13], which is

$$\begin{aligned}\left(\sum_{j=1}^{d_z}\hat{H}_{a,j,j}R_{j,j}\right)\left(\sum_{j=1}^{d_z}\hat{H}_{b,j,j}R_{j,j}\right) = \sum_{k=1}^{d_z}\hat{H}_{a,j,j}\hat{H}_{b,j,j}R_{j,j}^2 \\ +\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\bar{\delta}_{j,k}\hat{H}_{a,j,j}H_{b,k,k}R_{j,j}R_{k,k}\end{aligned} \tag{61}$$

to get

$$\begin{aligned} R_{ab}^{2,\text{debiased}} =& \sum_{j=1}^{d_z}\hat{J}_{a,j}\hat{J}_{b,j}R_{j,j}+\frac{3}{2}\sum_{j=1}^{d_z}\hat{H}_{a,j,j}\hat{H}_{b,j,j}R_{j,j}^2 \\ &+\frac{1}{2}\sum_{j=1}^{dz}\sum_{k=1}^{dz}\bar{\delta}_{j,k}\hat{H}_{a,j,k}\hat{H}_{b,j,k}R_{j,j}R_{k,k} \\ &+\sum_{j=1}^{d_z}\sum_{k=1}^{d_z}\bar{\delta}_{j,k}\hat{H}_{a,j,j}H_{b,k,k}R_{j,j}R_{k,k}\end{aligned} \tag{62}$$

which is the same as Equation 41 in [13].

IV. Simulation Examples

The second order algorithm of Marom [13] works well most of the time when given correlated measurement errors. Here, we consider two somewhat extreme examples to demonstrate-when Marom's algorithm tends to break down compared to the algorithms derived in this paper.

A. Debiased 2D Polar Measurement Conversion

Simple monostatic polar measurement conversion is a good baseline approach, as debiased conversion methods that ignore measurement cross terms, such as [10] exist in the literature. Here, a 2D Cartesian location $\mathbf{x}$ given a one-way range r and an azimuthal angle θ is

$$\mathbf{x} = \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta)\end{bmatrix} \tag{63}$$

The matrix of first derivatives of the conversion is

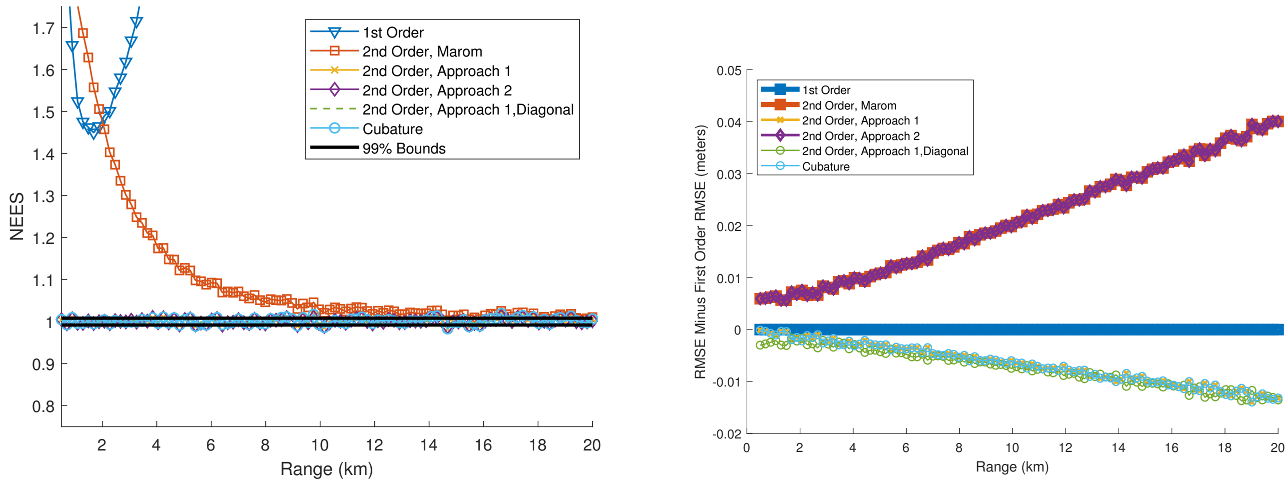
$$\mathbf{J} = \begin{bmatrix}\cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta)\end{bmatrix} \tag{64}$$

where the rows correspond to the elements of the Cartesian position $\mathbf{x}$ and the columns are the derivatives with respect to r and θ in that order. This is consistent with the notation of (10). Keeping with the format of (17), the matrix of second derivatives of the first component of $\mathbf{x}$ is component is

$$\mathbf{H}_{1,:,:} = \begin{bmatrix} 0 & -\sin(\theta) \\ -\sin(\theta) & -r\cos(\theta)\end{bmatrix} \tag{65}$$

and the matrix of second derivatives of the second component of $\mathbf{x}$ is

$$\mathbf{H}_{2,:,:} = \begin{bmatrix} 0 & \cos(\theta) \\ \cos(\theta) & -r\sin(\theta)\end{bmatrix} \tag{66}$$



(a) NEES (b) RMSE

Fig. 1. The NEES and RMSE as a function of the one-way range of the target for the correlated polar measurement conversion scenario. The first-order and Marom's method are not always NEES consistent, whereas the other algorithms are. In terms of RMSE, Maron's method and Approach 2 actually worsen the RMSE, whereas the other techniques improve the RMSE.

In this example, a target located at $\theta = 0$ at varying ranges is considered. The covariance matrix of the measurement is taken to be

$$\mathbf{R} = \begin{bmatrix} \sigma_r^2 & \rho\sigma_r\sigma_\theta \\ \rho\sigma_r\sigma_\theta & \sigma_\theta^2 \end{bmatrix} \tag{67}$$

with $\sigma_r = 20\,\mathrm{m}$, $\sigma_\theta = \frac{\pi}{180}\,\mathrm{rad}$, and $\rho = 0.999$, which is a high degree of correlation. Figure 1 shows the performance of the different algorithms averaged over $100,000$ Monte Carlo runs. The algorithms considered are:

- The first-order algorithm of (II-A).
- The second-order algorithm of Marom for general covariance matrices in Equations (43) and (50) of [12], which is derived without all of the cross terms in (22).
- Approach 1 for second order conversion of Section II, where the gradient approximation is performed in the Taylor series prior to deriving the estimator.
- Approach 2 for second order conversion of Section II, where the gradient approximation is performed after deriving the estimator assuming perfect gradients.
- Approach 1 for second order conversion, simplified assuming a diagonal covariance matrix, as derived in Section III.
- Fifth-order cubature integration, which is described in Section 8 of [4].

The normalized estimation error squared (NEES), which is a measure of the accuracy of the covariance matrix to the actual estimation and should be near 1 [4], is considered along with the root mean squared error (RMSE). It can be seen that the only algorithm that had a lower RMSE than the first order algorithm and also remains consistent all the time is Approach 1, with not much difference between consideration for the off-diagonal terms and ignoring the off-diagonal terms.

B. A Debiased Lorentz Transformation

An example of a transformation that is not traditionally considered for debiased estimation is a Lorentz transformation. An event in an inertial reference coordinate system is assigned a time t and a 3D position $\mathbf{t} = [x, y, z]'$ represented as a vector state $\mathbf{x} = [ct, x, y, z]'$, where c is the speed of light in a vacuum. Next, another inertial coordinate system that is moving with velocity $\mathbf{v}$ with respect to the reference system is considered. The timescale in the new system is $\tilde{t}$ and a 3D position in the new system is $\tilde{\mathbf{t}} = [\tilde{x}, \tilde{y}, \tilde{z}]'$ with the stacked state being $\tilde{\mathbf{x}}$. At time $t = 0$, it is assumed that $t = \tilde{t} = 0$ and that the origins of both coordinate systems coincide. The symmetric Lorentz transfomation from special relativity expresses how a position and time measurement in the reference coordinate system translates into a position and time in the moving coordinate system. Under Newtonian mechanics, the two coordinate systems would relate as

$$\tilde{t} = t \qquad \tilde{\mathbf{t}} = \mathbf{t} - \mathbf{v}t \tag{68}$$

because the origin point of the moving system is constantly changing. However, under special relativity, the two coordinate systems are related via a symmetric Lorentz transformation [11, Ch. 1.2]

$$\tilde{\mathbf{x}} = \mathbf{L}\mathbf{x} \tag{69}$$

$$\mathbf{L} = \begin{bmatrix} \gamma & -\frac{\gamma}{c}\mathbf{v}' \\ -\frac{\gamma}{c}\mathbf{v} & \mathbf{I} + (\gamma - 1)\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2} \end{bmatrix} \tag{70}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \tag{71}$$

There also exists an asymmetric Lorentz transformation [11, Ch. 1.2], where the time component of $\mathbf{x}$ is not multiplied by c. The real, symmetric form is used here, since it appears to offer more numeric stability in some instances. We would like to find a debiased conversion from $\mathbf{x}$ to $\tilde{\mathbf{x}}$ when given the measurement

$$\hat{\mathbf{p}} = \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{v}} \end{bmatrix} \sim \mathcal{N}\left\{\mathbf{p}, \mathbf{R}\right\} \tag{72}$$

where $\hat{\mathbf{p}}$ is a noisy measurement of time, position, and velocity (given as $\mathbf{p}$), in the reference coordinate system. In the event that time is assumed to be known exactly, the first row and column of $\mathbf{R}$ can be set to zero.

To perform the debiased measurement conversions of this paper, first and second derivatives are needed. The first derivatives of $\tilde{\mathbf{x}}$ with respect to component i of $\mathbf{p}$, p_i is simply

$$\frac{\partial \tilde{\mathbf{x}}}{\partial p_i} = \mathbf{L}\mathbf{e}_i + \left(\frac{\partial}{\partial p_i}\mathbf{L}\right)\mathbf{x} \tag{73}$$

where $\mathbf{e}_i$ is a 4×1 vector of all zeros with a single 1 in the ith spot if $i \leq 4$ and is a vector of all zeros if $i > 4$. Similarly, the second derivatives with $j \neq i$ are

$$\frac{\partial^2 \tilde{\mathbf{x}}}{\partial p_i^2} = 2\left(\frac{\partial}{\partial p_i}\mathbf{L}\right)\mathbf{e}_i + \left(\frac{\partial^2}{\partial p_i^2}\mathbf{L}\right)\mathbf{x} \tag{74}$$

$$\frac{\partial^2 \tilde{\mathbf{x}}}{\partial p_i \partial p_j} = \left(\frac{\partial}{\partial p_j}\mathbf{L}\right)\mathbf{e}_i + \left(\frac{\partial^2}{\partial p_i \partial p_j}\mathbf{L}\right)\mathbf{x} + \left(\frac{\partial}{\partial p_i}\mathbf{L}\right)\mathbf{e}_j \tag{75}$$

The first derivatives of $\mathbf{L}$ with respect to the non-velocity components can be written as

$$\frac{\partial}{\partial (ct)}\mathbf{L} = \frac{\partial}{\partial x}\mathbf{L} = \frac{\partial}{\partial y}\mathbf{L} = \frac{\partial}{\partial z}\mathbf{L} = \mathbf{0} \tag{76}$$

with all second derivatives involving any of those components also being zero. When considering velocity components $\mathbf{v} = [v_1, v_2, v_3]'$, the partial derivatives are

$$\frac{\partial}{\partial v_i}\mathbf{L} = \begin{bmatrix} \frac{\partial \gamma}{\partial v_i} & -\frac{\partial \gamma}{\partial v_i} - \frac{\frac{\partial \gamma}{\partial v_i}}{c}\mathbf{v}' - \frac{\gamma}{c}\mathbf{e}_i' \\ -\frac{\frac{\partial \gamma}{\partial v_i}}{c}\mathbf{v}' - \frac{\gamma}{c}\mathbf{e}_i' & \frac{\partial \gamma}{\partial v_i}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} + (\gamma - 1)\frac{\partial}{\partial v_i}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} \end{bmatrix} \tag{77}$$

$$\frac{\partial}{\partial v_i}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} = \frac{\mathbf{e}_i\mathbf{v}' + \mathbf{v}\mathbf{e}_i'}{\|\mathbf{v}\|^2} - 2\mathbf{v}\mathbf{v}'\frac{v_i}{\|\mathbf{v}\|^4} \tag{78}$$

$$\frac{\partial \gamma}{\partial v_i} = \frac{c v_i}{\left(c^2 - \|\mathbf{v}\|^2\right)^{\frac{3}{2}}} \tag{79}$$

The second derivatives of $\mathbf{L}$ with respect to the velocity components can be written

$$\frac{\partial^2}{\partial v_i^2}\mathbf{L} = \begin{bmatrix} \frac{\partial^2 \gamma}{\partial v_i^2} & (\mathbf{m}_{21}^s)' \\ \mathbf{m}_{21}^s & \mathbf{m}_{22}^s \end{bmatrix} \quad \frac{\partial^2}{\partial v_i \partial v_j}\mathbf{L} = \begin{bmatrix} \frac{\partial^2 \gamma}{\partial v_i \partial v_j} & \mathbf{m}_{21}' \\ \mathbf{m}_{21} & \mathbf{m}_{22} \end{bmatrix} \tag{80}$$

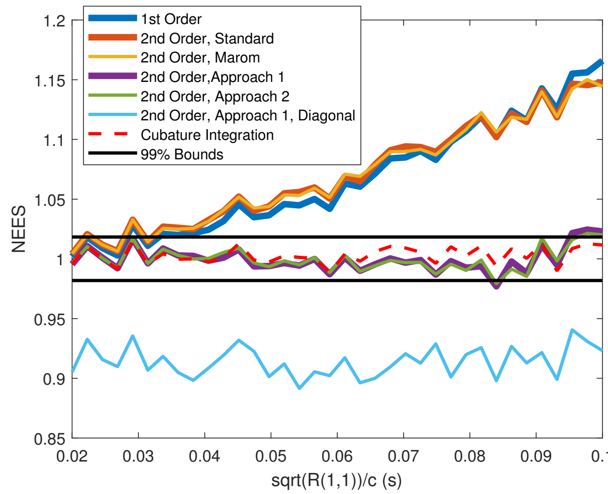


Fig. 2. The NEES performance of various algorithms in the highly correlated Lorentz transformation example. Cubature integration and the two second-order algorithms of this paper that account for covariance cross terms remain consistent, whereas the other techniques become inconsistent as the noise of the time component of the covariance matrix increases.

$$\mathbf{m}_{21}^s = -\frac{\frac{\partial^2 \gamma}{\partial v_i^2}}{c}\mathbf{v} - 2\frac{\frac{\partial \gamma}{\partial v_i}}{c}\mathbf{e}_i \tag{81}$$

$$\mathbf{m}_{22}^s = \frac{\partial^2 \gamma}{\partial v_i^2}\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2} + 2\frac{\partial \gamma}{\partial v_i}\frac{\partial}{\partial v_i}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} + (\gamma - 1)\frac{\partial^2}{\partial v_i^2}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} \tag{82}$$

$$\mathbf{m}_{21} = -\frac{\frac{\partial^2 \gamma}{\partial v_i \partial v_j}}{c}\mathbf{v}' - \frac{\frac{\partial \gamma}{\partial v_i}}{c}\mathbf{e}_j' - \frac{\frac{\partial \gamma}{\partial v_j}}{c}\mathbf{e}_i' \tag{83}$$

$$\mathbf{m}_{22} = \frac{\partial^2 \gamma}{\partial v_i \partial v_j}\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2} + \frac{\partial \gamma}{\partial v_i}\frac{\partial}{\partial v_j}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} + \frac{\partial \gamma}{\partial v_j}\frac{\partial}{\partial v_i}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} + (\gamma - 1)\frac{\partial^2}{\partial v_i \partial v_j}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} \tag{84}$$

$$\frac{\partial^2}{\partial v_i^2}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} = \frac{2\mathbf{e}_i\mathbf{e}_i'}{\|\mathbf{v}\|^2} - \frac{4v_i\left(\mathbf{e}_i\mathbf{v}' + \mathbf{v}\mathbf{e}_i'\right)}{\|\mathbf{v}\|^4} - 2\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^4} + 8\mathbf{v}\mathbf{v}'\frac{v_i^2}{\|\mathbf{v}\|^6} \tag{85}$$

$$\frac{\partial^2}{\partial v_i \partial v_j}\left\{\frac{\mathbf{v}\mathbf{v}'}{\|\mathbf{v}\|^2}\right\} = \frac{\mathbf{e}_i\mathbf{e}_j' + \mathbf{e}_j\mathbf{e}_i'}{\|\mathbf{v}\|^2} - \frac{2v_j\left(\mathbf{e}_i\mathbf{v}' + \mathbf{v}\mathbf{e}_i'\right)}{\|\mathbf{v}\|^4} - \frac{2v_i\left(\mathbf{e}_j\mathbf{v}' + \mathbf{v}\mathbf{e}_j'\right)}{\|\mathbf{v}\|^4} + 8\mathbf{v}\mathbf{v}'\frac{v_i v_j}{\|\mathbf{v}\|^6} \tag{86}$$

$$\frac{\partial^2 \gamma}{\partial v_i^2} = \frac{c\left(c^2 + 3v_i^2 - \|\mathbf{v}\|^2\right)}{\left(c^2 - \|\mathbf{v}\|^2\right)^{\frac{5}{2}}} \tag{87}$$

$$\frac{\partial^2 \gamma}{\partial v_i \partial v_j} = \frac{3c v_i v_j}{\left(c^2 - \|\mathbf{v}\|^2\right)^{\frac{5}{2}}} \tag{88}$$

As a simulation example, for a stacked state $\mathbf{p}$, the diagonal elements of the 7×7 covariance matrix are taken to be $\sigma_x^2 = 10^2\,\mathrm{m}^2$ for all of the position components , $\sigma_v^2 = 10^2\,(\mathrm{m/s})^2$ for all velocity components and σ_t^2 for the time component, where σ_t^2 is varied between $\left(2 \times 10^{-2}c\right)^2\,\mathrm{m}^2$ and $\left(10^{-1}c\right)^2\,\mathrm{m}^2$ (the units are meters after $c = 299{,}792{,}458\,\mathrm{m/s}$ is multiplied by a time in seconds). The element in every cross-diagonal term of the matrix is ρ times the product of the square root of the diagonals with $\rho = 0.99$. This is a highly correlated scenario.

For the conversion, the true time and position vector as well as the velocity vector of the moving inertial coordinate system are taken to be

$$\mathbf{x}=\begin{bmatrix}2c\,\mathrm{m}\\ 2\times 10^4\,\mathrm{m}\\ 1\times 10^4\,\mathrm{m}\\ -1\times 10^3\,\mathrm{m}\end{bmatrix}\qquad \mathbf{v}=\begin{bmatrix}-3/\sqrt{38}\\ 2/\sqrt{38}\\ 5/\sqrt{38}\end{bmatrix}0.99c \tag{89}$$

where c is the speed of light in a vacuum. Figure 2 shows the NEES of multiple algorithms averaged over $10,000$ Monte Carlo runs along with the 99% chi-squared confidence bounds. The two second-order approaches in this paper remain consistent as does fifth-order cubature integration. On the other hand, the first-order algorithm, the standard second order algorithm, which just computes the Jacobian and Hessian using noisy measurements, and Marom's algorithm, which omits some cross terms, all lose consistency as the standard deviation of the time component increase. Also shown is the first second order algorithm of this paper when assuming that the covariance matrix is diagonal. That algorithm is overly pessimistic the entire time. The RMSE is not plotted, but is not significantly different between the algorithms.

Consequently, this example demonstrates that the algorithms can improve the consistency of measurement conversion scenarios that go beyond those traditionally considered for target tracking.

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V. Conclusions

This work derived two new second-order approaches for approximating the first two moments of a Gaussian random vector that goes through a nonlinear transformation, where the covariance matrix $\mathbf{R}$ can have non-zero cross terms. Unlike similar work by Marom in [14], [13], and [12], all of the derivatives of a first-order Taylor series expansion of the Jacobian were taken into account. Over both scenarios considered here, the overall best algorithm was the second order Approach 1 developed in this paper, though Approach 2 also performed well. In both scenarios, Marom's method exhibited covariance inconsistency. In the first scenario, it also exhibit slightly inferior RMSE performance. In the specific examples considered, the performance was comparable to fifth-order cubature integration. However, the computational complexity can be notably lower than cubature integration as the dimensionality of the problem increases. In all, the algorithms of this paper should be applicable to any continuous multivariate transformation for which one can obtain first and second derivatives of the estimate with respect to the inputs of the estimator.

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